Solving non-linear HJB PDE for robust economic-climate models

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What We Do

- well studied: discrete-time dynamic programming methods
- alternative: small-noise expansion in state space
- new: numerical solution of robust stochastic macroeconomic models in continuous time with optimal control
- benefits: simpler computational procedure; technique is very general, allows robustness analysis over variety of fundamentally different models
- open issues: existence and uniqueness; better numerical methods for computing

General formulation: non-linear max-min HJB PDE

$$
\begin{aligned}
0 &= -\frac{\partial W(X_1, ..., X_l; t)}{\partial t} \\
&= \max_{C_1,\ldots,C_m} \min_{\Lambda_1,\ldots,\Lambda_n} \Bigg[u(C_1, ..., C_m) + \sum_{k=1}^{n} \frac{1}{2\theta_k} \Lambda_k^2 \\
&\qquad + \sum_{i=1}^{l} \frac{\partial W(X_1, ..., X_l; t)}{\partial X_i} f_i(X_1, ..., X_l; C_1, ..., C_m; \Lambda_1, ..., \Lambda_n) \\
&\qquad + \sum_{i=1}^{l} \frac{\partial^2 W(X_1, ...X_l; t)}{\partial X_i^2} g_i(X_1, ..., X_l; C_1, ..., C_m; \Lambda_1, ..., \Lambda_n) \\
&\qquad + W(X_1, ..., X_l; t) h(X_1, ...X_l; C_1, ..., C_m; \Lambda_1, ..., \Lambda_n) \Bigg]
\end{aligned}
$$

General formulation: optimal controls and minimizers

Controls (maximizers)

$$C_j^* = a_j(X_1, ..., X_I) * F_j\left(\frac{\partial W(X_1, ..., X_I; t)}{\partial X_1}, ..., \frac{\partial W(X_1, ..., X_I; t)}{\partial X_I}\right)$$

Robustness channels (minimizers)

$$\Lambda_k^* = b_k(X_1, ..., X_I) * G_k\left(\frac{\partial W(X_1, ..., X_I; t)}{\partial X_1}, ..., \frac{\partial W(X_1, ..., X_I; t)}{\partial X_I}\right)$$

Convergent approximation: numerical methods

- theory of viscosity solutions by Crandall and Lions (84,92,96)
- Barles-Souganidis (91) proof: monotone finite-difference (FD) method can converge to a well-behaved unique solution of the fully nonlinear Hamilton-Jacobi-Bellman (HJB) equation with no explicit boundary conditions
- Oberman (06): Convergent numerical schemes for PDE are constructed and implemented
- we apply a FD numerical approach to solve Ramsey-type economic growth models with carbon-climate response (CCR) and Hansen-Sargent robustness channels

Hennlock's two-sector model: HJB PDE

$$
\begin{aligned}
0 &= -\frac{\partial W(K, E_k, A_K, A_E, M, T; t)}{\partial t} \\
&= \max_{C,G,q,r,s} \min_{\Lambda_0} \left[\frac{1}{1-\eta} \left[(1-\omega) C^{\frac{\sigma-1}{\sigma}} + \omega G^{\frac{\sigma-1}{\sigma}} \right]^{\frac{(1-\eta)\sigma}{\sigma-1}} \right. \\
&\qquad + \frac{\theta_0}{2} \Lambda_0^2 - \rho * W + \frac{\partial W}{\partial K} \left[\left(v_K + (1-r) A_K^{\tau} \right) K^{\alpha} - cq^2 - C - \delta K \right] \\
&\quad + \frac{\partial W}{\partial E_K} \left[\left(v_E + (1-s) A_E^{\Psi} \right) E^{\phi} - \frac{1}{\kappa} E - \Theta(T - T_0) E^{\phi} - \pi G \right] + \frac{\partial W}{\partial A_K} \left[\nu (r A_K^{\tau} K^{\alpha})^{\tau} A_K^{1-\tau} - \delta_{A_K} A_K \right] \\
&\qquad + \frac{\partial W}{\partial A_E} \left[\beta (s A_E^{\Psi} E^{\phi})^{\Psi} A_E^{1-\Psi} - \delta_{A_E} A_E \right] + \frac{\partial W}{\partial M} \left[\epsilon (\chi K^{\alpha} - \mu q) - \Omega M \right] \\
&\quad \left. + \frac{\partial W}{\partial T} \frac{1}{\tau_1} \left[\Lambda_0 \frac{\sigma_v \sqrt{M/M_0}}{\sqrt{2}\gamma} + \lambda_0 \frac{M/M_0}{2\gamma} - \lambda_1 T - \frac{\tau_3}{\tau_2} (T - \tilde{T}) \right] + \frac{1}{2} \frac{\partial^2 W}{\partial T^2} \sigma_v^2 M + \frac{h}{\tau_3} \frac{\tau_1}{\tau_2} (T - \tilde{T}) \right]
\end{aligned}
$$

Solution of PDE (analytical by Hennlock, valid only for the special choice of structural parameters)

$$
W(K, E_k, A_K, A_E, M, T, \tilde{T}) = e^{-\rho t} (a K^{1-\alpha} + b A_K^{\tau} + d E^{1-\phi} + e A_E^{\psi} + fM + gT + h\tilde{T} + \kappa)
$$

Hennlock's two-sector model: controls and minimizers

Controls (generally true, no further simplification by Hennlock is used)
carbon-intensive consumption feedback:

$$C^*(K) = \left(\frac{1-\omega}{\frac{\partial W}{\partial K}} \right)^{1/\eta}$$

carbon-neutral consumption feedback:

$$G^*(E) = \left(\frac{\omega}{\pi \frac{\partial W}{\partial E}} \right)^{1/\eta}$$

abatement feedback:

$$q^*(K) = -\mu \frac{\epsilon}{2c} \frac{\partial W}{\partial M} / \frac{\partial W}{\partial K}$$

carbon-intensive research effort feedback:

$$r^*(A_K, K) = \frac{A_K^{1-\tau} (\nu \tau)^{\frac{1}{1-\tau}}}{K^{\alpha}} \left[\frac{\partial W}{\partial A_K} / \frac{\partial W}{\partial K} \right]^{\frac{1}{1-\tau}}$$

carbon-neutral research effort feedback:

$$s^*(A_E, E) = \frac{A_E^{1-\Psi} (\beta \Psi)^{\frac{1}{1-\Psi}}}{E^{\phi}} \left[\frac{\partial W}{\partial A_E} / \frac{\partial W}{\partial E} \right]^{\frac{1}{1-\Psi}}$$

minimizer feedback:

$$\Lambda^*(M) = -\frac{\partial W}{\partial T} \frac{\sigma_v \sqrt{M/M_0}}{\theta_0 \tau_1 \gamma \sqrt{2}}$$

Finite-differencing schemes and numerical solutions

1. explicit upwind scheme is used
2. no explicit boundaries are imposed, piecewise cubic Hermite interpolation from the interior of the grid is used instead for derivatives at the boundaries
3. SIMD (single instruction, multiple data) vector processing optimized multi-dimensional arrays with Kronecker tensor products are used throughout to make computations highly efficient
4. iterations result in a stable (but somewhat slow) linear convergence throughout, results can be improved to higher precision if required, cold start can be used to split the execution in batches

Finite-differencing schemes and numerical solutions

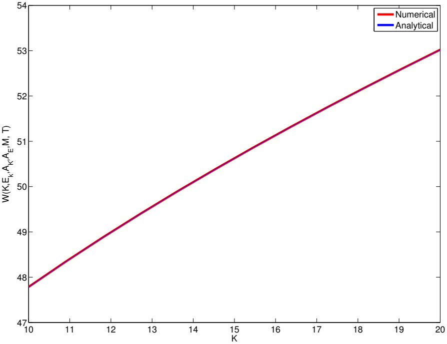

Figure : Hennlock HJB PDE: Value function

There is no discernible difference between analytical and numerical solutions for value function. There is also no difference between analytical and numerical solutions for policies.

Relaxing Hennlock's assumptions

Capital intensity to 0.3 - typical value in growth models, instead of 1/2.
Different CES elasticity of substitution.

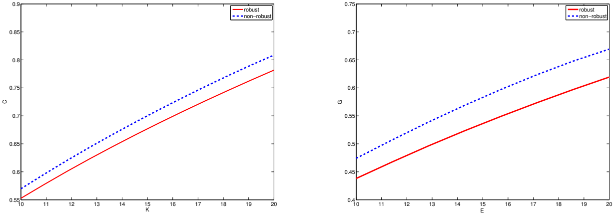

Figure : Policies: consumption of carbon intensive and carbon neutral goods, robust vs. non-robust

We see more curvature and larger difference in numerical solution between robust vs. non-robust specification

Anderson-Brock-Hansen-Sanstad (ABHS) robust economic-climate model

$$
\begin{aligned}
0 &= -\frac{\partial W(K, R, \hat{T}, \log A, \log D; t)}{\partial t} \\
&= \max_{C,E} \min_{G_T, G_a, G_d} \Bigg[\frac{1}{1-\gamma} C^{1-\gamma} + \frac{1}{2\theta}\left(G_T^2 + G_a^2 + G_d^2\right) - \rho * W \\
&\qquad + \frac{\partial W}{\partial K}\left[A \exp[-D(\overline{T}_0 + \hat{T} + \lambda(R_0 - R) - \dot{T})] K^{\alpha} E^{\nu} - C - \delta K\right] + \frac{\partial W}{\partial R}[-E] \\
&\qquad + \frac{\partial W}{\partial \hat{T}}\left[-\kappa_T \hat{T} - \sigma_T \sqrt{\epsilon} G_T\right] + \frac{\partial W}{\partial \log A}\left[\mu_{a0} - \mu_{aT}(\overline{T}_0 + \hat{T} + \lambda(R_0 - R) - \dot{T}) - \sigma_a \sqrt{\epsilon} G_a\right] \\
&\qquad + \frac{\partial W}{\partial \log D}\left[\kappa_d(\mu_d - \log D) - \sigma_d \sqrt{\epsilon} G_d\right] + \frac{1}{2}\frac{\partial^2 W}{\partial T^2}\sigma_T^2 \epsilon + \frac{1}{2}\frac{\partial^2 W}{\partial \log A^2}\sigma_a^2 \epsilon + \frac{1}{2}\frac{\partial^2 W}{\partial \log D^2}\sigma_d^2 \epsilon \Bigg]
\end{aligned}
$$

ABHS Controls

consumption feedback:

$$C^*(K, R, \hat{T}, \log A, \log D) = \left(\frac{1}{\frac{\partial W}{\partial K}} \right)^{1/\gamma}$$

energy feedback:

$$E^*(K, R, \hat{T}, \log A, \log D) = \frac{1}{\nu A \exp[-D(T_0 + \hat{T} + \lambda(R_0 - R) - \dot{T})] K^{\alpha}} \left(\frac{\frac{\partial W}{\partial R}}{\frac{\partial W}{\partial K}} \right)^{\frac{1}{\nu - 1}}$$

temperature distortion channel:

$$G_T^*(K, R, \hat{T}, \log A, \log D) = \theta \frac{\partial W}{\partial \hat{T}} \sigma_T \sqrt{\epsilon}$$

technology distortion channel:

$$G_a^*(K, R, \hat{T}, \log A, \log D) = \theta \frac{\partial W}{\partial \log A} \sigma_a \sqrt{\epsilon}$$

climate damage distortion channel:

$$G_d^*(K, R, \hat{T}, \log A, \log D) = \theta \frac{\partial W}{\partial \log D} \sigma_d \sqrt{\epsilon}$$

State constraints

There are additional state and choice constraints:

$$C, E, R, K \geq 0$$

In order to stay in feasible region we need to assure that

$$K_{i_k} \geq A_{i_a} \exp[-D_{i_d}(\overline{T}_0 + \hat{T}_{i_T} + \lambda(R_0 - R_{i_r}) - \dot{T})] K_{i_k}^{\alpha} E_{i_k,i_r,i_T,i_a,i_d}^{\nu} \\ - C_{i_k,i_r,i_T,i_a,i_d} - \delta K_{i_k}, \forall i_k, i_r, i_T, i_a, i_d$$

$$R_{i_r} \geq E_{i_k,i_r,i_T,i_a,i_d}, \forall i_k, i_r, i_T, i_a, i_d$$

Conclusion and future work

- robust stochastic PDE for macroeconomics models with climate coupling can be solved by free-boundary finite-difference methods
- numerical methods are relatively simple and general
- robust stochastic PDE approach allows to perform robustness analysis quickly a variety of fundamentally different macroeconomics models and uncover solutions for the entire state space

- Future work

  1. Pricing uncertainty over alternative horizons
  2. Building dynamic games between the private sector and the governments under alternative specifications of ambiguity
  3. Developing faster, more reliable finite-difference schemes
  4. Using data to calibrate or estimate models